2020 MAA AIME I

1. In $\triangle ABC$ with $AB = AC$, point D lies strictly between A and C on side $\overline{AC}$, and point E lies strictly between A and B on side $\overline{AB}$ such that $AE = ED = DB = BC$. The degree measure of $\angle ABC$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

   在 $\triangle ABC$ 中，$AB = AC$，点 D 严格位于边 $\overline{AC}$ 上的 A 和 C 之间，而点 E 严格位于边 $\overline{AB}$ 上的 A 和 B 之间，使得 $AE = ED = DB = BC$。$\angle ABC$ 的度数为 $\frac{m}{n}$，其中 m 和 n 是互质的正整数。求 $m+n$。

2. There is a unique positive real number x such that the three numbers $\log_8(2x)$, $\log_4 x$, and $\log_2 x$, in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

   存在唯一的正实数 x，使得三个数 $\log_8(2x)$、$\log_4 x$、$\log_2 x$ 按此顺序形成公比为正数的几何级数。数 x 可以写成 $\frac{m}{n}$，其中 m 和 n 是互质的正整数。求 $m+n$。

3. A positive integer N has base-eleven representation $\underline{a}\,\underline{b}\,\underline{c}$ and base-eight representation $\underline{1}\,\underline{b}\,\underline{c}\,\underline{a}$, where a, b, and c represent (not necessarily distinct) digits. Find the least such N expressed in base ten.

   正整数 N 的 11 进位制表示是 $\underline{a}\,\underline{b}\,\underline{c}$，8 进位制表示是 $\underline{1}\,\underline{b}\,\underline{c}\,\underline{a}$，其中 a、b 和 c 表示（不一定相异的）数字。找出这样的 N 中最小的数的十进制表示。

4. Let S be the set of positive integers N with the property that the last four digits of N are 2020, and when the last four digits are removed, the result is a divisor of N. For example, $42,020$ is in S because 4 is a divisor of $42,020$. Find the sum of all the digits of all the numbers in S. For example, the number $42,020$ contributes $4+2+0+2+0=8$ to this total.

设 S 是具有如下性质的正整数 N 的集合：N 的最后四位数字是 2020，并且当最后四位数字去掉后所得的数是 N 的约数。例如，42,020 在 S 中，因为 4 是 $42,020$ 的约数。求 S 中所有数的各个数字的总和。例如，数 $42,020$ 对这个总和的贡献为 $4+2+0+2+0=8$。

5. Six cards numbered 1 through 6 are to be lined up in a row. Find the number of arrangements of these six cards where one of the cards can be removed leaving the remaining five cards in either ascending or descending order.

六张写有数从 1 到 6 的卡片将排成一行。如果其中的一张卡片可以被移除，使得剩下的五张卡片是按升序或降序排列的，那么六张卡片这样的排列方式有多少种?

6. A flat board has a circular hole with radius 1 and a circular hole with radius 2 such that the distance between the centers of the two holes is 7. Two spheres with equal radii sit in the two holes such that the spheres are tangent to each other. The square of the radius of the spheres is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

平板上有一个半径为 1 的圆孔和一个半径为 2 的圆孔，两个圆孔的中心之间的距离为 7。两个半径相等的球体放置在两个孔上，并且彼此相切。球体半径的平方是 $\frac{m}{n}$，其中 m 和 n 是互质的正整数。求 $m+n$。

7. A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committee could have as few as 1 member or as many as 23 members. Let N be the number of such committees that can be formed. Find the sum of the prime numbers that divide N.

由 11 位男士和 12 位女士组成的俱乐部需要由其成员组成一个委员会，该委员会中的女性人数比男性人数多一。该委员会的成员最少有 1 位，最多有 23 位。假设 N 是可以成立的此类委员会的数目。求 N 的所有质约数之和。

8. A bug walks all day and sleeps all night. On the first day, it starts at point O, faces east, and walks a distance of 5 units due east. Each night the bug rotates 60° counterclockwise. Each day it walks in this new direction half as far as it walked the previous day. The bug gets arbitrarily close to point P. Then $OP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

虫子白天行走，夜晚睡觉。第一天，它从 O 点开始，然后向东走了 5 个单位。虫子每晚都会逆时针旋转 60°。每天，它都会沿新方向行走，行走的距离是前一天的一半。虫子会无限接近点 P。假设 $OP^2 = \frac{m}{n}$，其中 m 和 n 是互质的正整数。求 $m+n$。

9. Let S be the set of positive integer divisors of 20^9. Three numbers are chosen independently and at random with replacement from the set S and labeled a_1, a_2, and a_3 in the order they are chosen. The probability that both a_1 divides a_2 and a_2 divides a_3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

设 S 为 20^9 的正整数约数组成的集合。独立随机的从集合 S 中选择三个数，已经选择的数可以再选，记依次选出的数分别为 a_1，a_2 和 a_3。a_1 能够除尽 a_2 和 a_2 能够除尽 a_3 的概率为 $\frac{m}{n}$，其中 m 和 n 是互质的正整数。求 m。

10. Let m and n be positive integers satisfying the conditions

- $\gcd(m+n, 210) = 1$,
- m^m is a multiple of n^n, and
- m is not a multiple of n.

Find the least possible value of $m+n$.

设 m 和 n 为满足下述条件的正整数：

- $\gcd(m+n, 210) = 1$,
- m^m 是 n^n 的倍数，并且
- m 不是 n 的倍数。

求 $m+n$ 的最小可能值。

11. For integers a, b, c, and d, let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$. Find the number of ordered triples (a, b, c) of integers with absolute values not exceeding 10 for which there is an integer d such that $g(f(2)) = g(f(4)) = 0$.

对于整数 a，b，c 和 d，令 $f(x) = x^2 + ax + b$ 和 $g(x) = x^2 + cx + d$。求出绝对值不超过 10 的有序三元整数组 (a, b, c) 的个数，使得存在整数 d 使得 $g(f(2)) = g(f(4)) = 0$。

12. Let n be the least positive integer for which $149^n - 2^n$ is divisible by $3^3 \cdot 5^5 \cdot 7^7$. Find the number of positive integer divisors of n.

令 n 是使得 $149^n - 2^n$ 能被 $3^3 \cdot 5^5 \cdot 7^7$ 整除的最小正整数。求 n 的正整数约数的个数。

13. Point D lies on side $\overline{BC}$ of $\triangle ABC$ so that $\overline{AD}$ bisects $\angle BAC$. The perpendicular bisector of $\overline{AD}$ intersects the bisectors of $\angle ABC$ and $\angle ACB$ in points E and F, respectively. Given that $AB = 4$, $BC = 5$, and $CA = 6$, the area of $\triangle AEF$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m + n + p$.

点 D 位于 $\triangle ABC$ 的 $\overline{BC}$ 边上，$\overline{AD}$ 是 $\angle BAC$ 的角平分线。垂直并且平分 $\overline{AD}$ 的直线与 $\angle ABC$ 和 $\angle ACB$ 的角平分线分别相交于点 E 和 F。已知 $AB = 4$，$BC = 5$，并且 $CA = 6$，则 $\triangle AEF$ 的面积可以写为 $\frac{m\sqrt{n}}{p}$，其中 m 和 p 是相对质数的正整数，而 n 是不可被任何质数的平方整除的正整数。求 $m + n + p$。

14. Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a + b)^2$.

设 $P(x)$ 是复系数的二次多项式，其中 x^2 项的系数为 1。假设方程 $P(P(x)) = 0$ 有四个不同的解，$x = 3$，4，a，b。找出 $(a + b)^2$ 所有可能的值的总和。

15. Let $\triangle ABC$ be an acute triangle with circumcircle ω, and let H be the intersection of the altitudes of $\triangle ABC$. Suppose the tangent to the circumcircle of $\triangle HBC$ at H intersects ω at points X and Y with $HA = 3$, $HX = 2$, and $HY = 6$. The area of $\triangle ABC$ can be written as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

设 $\triangle ABC$ 是一个锐角三角形，其外接圆是 ω，H 是 $\triangle ABC$ 的高线的交点。假设 $\triangle HBC$ 的外接圆在 H 的切线与 ω 相交在点 X 和 Y，并且 $HA = 3$，$HX = 2$，$HY = 6$。$\triangle ABC$ 的面积可以写成 $m\sqrt{n}$，其中，m 和 n 是互质的正整数，n 不能被任何质数的平方整除。求 $m + n$。