2019 AIME I

1. Consider the integer

$$N = 9 + 99 + 999 + 9999 + \cdots + \underbrace{99\cdots99}_{321\text{digits}}.$$

Find the sum of the digits of N.

考慮整數

$$N = 9 + 99 + 999 + 9999 + \cdots + \underbrace{99\cdots99}_{321\text{個數字}}.$$

求N的各位數字之和。

2. Jenn randomly chooses a number J from $1,2,3,\ldots,19,20$. Bela then randomly chooses a number B from $1,2,3,\ldots,19,20$ distinct from J. The value of $B-J$ is at least 2 with a probability that can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Jenn從1，2，3，...，19，20中隨機選擇一個數J。隨後Bela從1，2，3，...，19，20中隨機選擇一個不同於J的數B，$B-J$的值至少為2的概率可以用$\frac{m}{n}$的形式表示，其中m和n是互質的正整數。求$m+n$。

3. In $\triangle PQR$, $PR=15$, $QR=20$, and $PQ=25$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, and points E and F lie on $\overline{PR}$, with $PA=QB=QC=RD=RE=PF=5$. Find the area of hexagon $ABCDEF$.

在$\triangle PQR$中，$PR=15$，$QR=20$，$PQ=25$。點A和B在$\overline{PQ}$上，點C和D在$\overline{QR}$上，點E和F在$\overline{PR}$上，並且$PA=QB=QC=RD=RE=PF=5$。求六邊形 $ABCDEF$的面積。

4. A soccer team has 22 available players. A fixed set of 11 players starts the game, while the other 11 are available as substitutes. During the game, the coach may make as many as 3 substitutions, where any one of the 11 players in the game is replaced by one of the substitutes. No player removed from the game may reenter the game, although a substitute entering the game may be replaced later. No two substitutions can happen at the same time. The players involved and the order of substitutions matter. Let n be the number of ways the coach can make substitutions during the game (including the possibility of making no substitutions). Find the remainder when n is divided by 1000.

一支足球隊有22個可用的球員。開始時，由確定的11名隊員參加比賽，而另外11名隊員為可用替補。在比賽期間，教練可以進行最多3次的替換，每次替換是指參賽的11名隊員中的某一人被一名替補隊員替換。下場的隊員不能再上場，但是替補隊員上場後可以再被替換下場。替換中涉及的球員和替換順序均需考慮。設n是教練在比賽期間可以進行替換的方式（包括不進行替換這種可能性）的總數。求n除以1000的餘數。

5. A moving particle starts at the point $(4,4)$ and moves until it hits one of the coordinate axes for the first time. When the particle is at the point (a,b), it moves at random to one of the points $(a-1,b)$, $(a,b-1)$, or $(a-1,b-1)$, each with probability $\frac{1}{3}$, independently of its previous moves. The probability that it will hit the coordinate axes at $(0,0)$ is $\frac{m}{3^n}$, where m and n are positive integers, and m is not divisible by 3. Find $m+n$.

   一個移動的粒子從點(4,4)開始移動，直到它第一次碰到某個坐標軸終止。當粒子位於點(a,b)時，它會隨機移動到$(a-1,b)$，$(a,b-1)$或$(a-1,b-1)$中的一點，每種情況的概率都是$\frac{1}{3}$，並且與之前的移動互相獨立。它達到坐標軸時是在(0,0)處的概率是$\frac{m}{3^n}$，其中m和n是正整數，而m不能被3除盡。求$m+n$。

6. In convex quadrilateral $KLMN$ side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN=65$, and $KL=28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO=8$. Find MO.

   在凸四邊形$KLMN$中，邊$\overline{MN}$垂直於對角線$\overline{KM}$，邊$\overline{KL}$垂直於對角線$\overline{LN}$，$MN=65$，並且$KL=28$。通過L垂直於邊$\overline{KN}$的直線與對角線$\overline{KM}$相交於O，並且$KO=8$。求MO。

7. There are positive integers x and y that satisfy the system of equations

$$\log_{10} x + 2\log_{10}(\gcd(x,y)) = 60,$$

$$\log_{10} y + 2\log_{10}(\text{lcm}(x,y)) = 570.$$

   Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y. Find $3m+2n$.

   有正整數x和y滿足方程組

$$\log_{10} x + 2\log_{10}(\gcd(x,y)) = 60，$$

$$\log_{10} y + 2\log_{10}(\text{lcm}(x,y)) = 570。$$

   設m是x的素因數分解中的（不一定是不同的）素數因子的數目，並且n是y的素因數分解中的（不一定是不同的）素數因子的數目。求$3m+2n$。

8. Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Then $\sin^{12} x + \cos^{12} x = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

   設實數x滿足$\sin^{10} x + \cos^{10} x = \frac{11}{36}$。設$\sin^{12} x + \cos^{12} x = \frac{m}{n}$，其中$m$和$n$是互質的正整數。求$m+n$。

9. Let $\tau(n)$ denote the number of positive integer divisors of n. Find the sum of the six least positive integers n that are solutions to $\tau(n)+\tau(n+1)=7$.

   設$\tau(n)$表示n的正整數約數的個數。求滿足 $\tau(n)+\tau(n+1)=7$的最小的六個正整數n的總和。

10. For distinct complex numbers $z_1, z_2, \ldots, z_{673}$, the polynomial

$$(x-z_1)^3(x-z_2)^3\cdots(x-z_{673})^3$$

can be expressed as $x^{2019}+20x^{2018}+19x^{2017}+g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left|\sum_{1\le j<k\le 673} z_j z_k\right|$$

can he expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

對於不同的複數z_1，z_2，…， z_{673}，多項式

$$(x-z_1)^3(x-z_2)^3\cdots(x-z_{673})^3$$

可以表示成 $x^{2019}+20x^{2018}+19x^{2017}+g(x)$, 其中$g(x)$是一個次數最高為2016的複係數多項式。

$$\left|\sum_{1\le j<k\le 673} z_j z_k\right|$$

的值可以表達成 $\frac{m}{n}$的形式，其中m和n是互質的正整數。求$m+n$。

11. In $\triangle ABC$, the sides have integer lengths and $AB=AC$. Circle ω has its center at the incenter of $\triangle ABC$. An *excircle* of $\triangle ABC$ is a circle in the exterior of $\triangle ABC$ that is tangent to one side of the triangle and tangent to the extensions of the other two sides. Suppose that the excircle tangent to $\overline{BC}$ is internally tangent to ω, and the other two excircles are both externally tangent to ω. Find the minimum possible value of the perimeter of $\triangle ABC$.

在$\triangle ABC$中，各邊長度均為整數，$AB=AC$。圓ω的圓心是$\triangle ABC$的內心。 $\triangle ABC$的旁切圓是 $\triangle ABC$外部的一個圓，它與三角形的一邊相切，與另外兩邊的延長線相切。假設與$\overline{BC}$相切的旁切圓與ω內切，而另外兩個旁切圓與ω外切。求$\triangle ABC$週長的最小可能值。

12. Given $f(z)=z^2-19z$, there are complex numbers z with the property that z, $f(z)$, and $f(f(z))$ are the vertices of a right triangle in the complex plane with a right angle at $f(z)$. There are positive integers m and n such that one such value of z is $m+\sqrt{n}+11i$. Find $m+n$.

設$f(z)=z^2-19z$，有復數z具有性質：z，$f(z)$和$f(f(z))$是復平面中直角三角形的頂點，直角頂點在$f(z)$。存在正整數m和n，使得一個這樣z的值是$m+\sqrt{n}+11i$。求$m+n$。

13. Triangle ABC has side lengths $AB=4$, $BC=5$, and $CA=6$. Points D and E are on ray AB with $AB<AD<AE$. The point $F\neq C$ is a point of intersection of the circumcircles of $\triangle ACD$ and $\triangle EBC$ satisfying $DF=2$ and $EF=7$. Then BE can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a,b,c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.

三角形ABC的邊長為$AB=4$，$BC=5$，$CA=6$。點D和E在射線AB上並且$AB<AD<AE$。點$F\neq C$是$\triangle ACD$和$\triangle EBC$的外接圓的交點，滿足$DF=2$和$EF=7$。 BE可以表示為$\frac{a+b\sqrt{c}}{d}$的形式，其中a，b，c和d是正整數,使得a和d互質，並且c不能被任何素數的平方整除。求$a+b+c+d$。

14. Find the least odd prime factor of 2019^8+1.

求2019^8+1的最小的奇數素因子。

15. Let $\overline{AB}$ be a chord of circle ω, and let P be a point on chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP=5$, $PB=3$, $XY=11$, and $PQ^2=\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

設$\overline{AB}$是圓ω的一條弦，並且P是弦$\overline{AB}$上的一個點。圓ω_1通過A和P，並且與ω內切。圓ω_2通過B和P，並且與ω內切。圓ω_1和ω_2相交於點P和Q。直線PQ與ω相交於X和Y。假設$AP=5$，$PB=3$，$XY=11$，並且$PQ^2=\frac{m}{n}$，其中m和n是互質的正整數。求$m+n$。